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LETTER TO THE EDITOR

On the spin–fermion connection

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Abstract
Jordan–Wigner-type transformations connecting the spin-3/2 operators and two types of fermion are derived. The general condition of fermionizability of spins is obtained. Discordances in the results of a previous attempt to generalize the Jordan–Wigner transformation for all spins (Batista and Ortiz 2001 *Phys. Rev. Lett.* **86** 1082) are pointed out. After introducing clarity, a new interpretation is given for these results.

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1. Introduction

Variable exchanges are very often used during the solution of different problems. Such transformations are carried out in both classical and quantum mechanics. Sometimes they essentially simplify the analysis of the problem under consideration.

In the quantum many-body theory there is a set of variable exchanges transforming a spin system into a particle system. The most well known and widely used among these are the Holstein–Primakoff transformation [1], the Schwinger boson representation [2] and the Jordan–Wigner transformations [3] (see also [4, 5]). The first two of these map one spin on to a bosonic system. However, due to the fact that the phase space of the spin is finite-dimensional but the phase space of the bosonic system is infinite-dimensional, the next main difficulty arises. It is necessary to take into account some local constraints fixing or restricting the number of bosons in the particle system. This circumstance limits the use of these methods.

The Jordan–Wigner transformations are quite another matter. These establish a connection between one-dimensional (1D) lattice spin-1/2 and spinless fermions on the same lattice; they are invertible and do not require a constraint. In spite of their complicated form, they have led to several exact results. The most famous applications of the Jordan–Wigner transformations are the exact solution of the spin-1/2 XY-chain by Lieb *et al* [6] and the very effective free fermion method developed by the same researchers to exactly calculate the thermodynamic quantities of the two-dimensional (2D) Ising model [7]. Quite recently, these transformations were generalized for 2D [8] and for $D > 2$ [9] spin-1/2 systems.

However, in nature, all spins up to 15/2 exist [10]. It is very interesting and important to understand the behaviour of spins higher than 1/2 systems. In spite of the great progress achieved with the help of the above-mentioned spin–boson transformations and other methods, many questions so far remain unanswered, for example, the problem of the Haldane gap [11]. It is extremely desirable to obtain more exact solutions, sometimes yielding unexpected results. Thus, it would be very useful to obtain some new representations for operators of higher spins, free of the shortcomings of bosonic representations. To obtain this aim, it is necessary to use particles with finite-dimensional phase space, i.e. fermions. Is this possible for all higher spins? What use can we derive from Jordan–Wigner-type transformations obtained in this way? To answer these questions, first of all we must try to obtain such relations between spins and fermions. Only after this, will these things become clear. Recently, an attempt to generalize the Jordan–Wigner transformations was undertaken by Batista and Ortiz [12]. Unfortunately, their results contain serious discordances. Below we shall demonstrate this and, after introducing clarity, we shall give another interpretation for their formulae. Then we shall construct explicitly true Jordan–Wigner-type transformations between spin-3/2 and fermions of two types. In conclusion, we shall formulate the general condition of fermionizability of spins in the Jordan and Wigner manner.

2. Clarification of the results of Batista and Ortiz

We shall find discordances in the Batista and Ortiz results considering two examples: (1) for integer spin; (2) for half integer spin.

To begin, we shall examine the transformations of Batista and Ortiz for spin-1. We use the language of usual Fermi operators of creation and annihilation but not of Hubbard operators, as in [12]. Also, we omit in this section the site indices and the string operators as non-essential for our consideration. Then the Batista–Ortiz transformations can be written as

$$S^+ = \sqrt{2}\big((1-n_1)c_2^\dagger + (1-n_2)c_1\big) \tag{1}$$

$$S^- = \sqrt{2}\big((1-n_1)c_2 + (1-n_2)c_1^\dagger\big). \tag{2}$$

Here, $S^\pm = S^x \pm \mathrm{i}S^y$, S^x, S^y and S^z are the operators of spin components, and the next commutative relations must be fulfilled

$$[S^+, S^-] = 2S^z \qquad [S^z, S^\pm] = \pm S^\pm \qquad \{S^+, S^-\} = 2S(S+1) - 2(S^z)^2 \tag{3}$$

where S is the value of the spin; in our case $S = 1$. $c_{1,2}$ and $c_{1,2}^\dagger$ are the annihilation and creation operators for fermions of the first (labelled 1) and second (labelled 2) types with the next anticommutative relations

$$\{c_\alpha, c_\beta\} = 0 \qquad \big\{c_\alpha^\dagger, c_\beta^\dagger\big\} = 0 \qquad \big\{c_\alpha, c_\beta^\dagger\big\} = \delta_{\alpha\beta} \quad (\alpha, \beta = 1, 2) \tag{4}$$

and $n_\alpha = c_\alpha^\dagger c_\alpha$.

From equations (1) and (2) it follows that

$$S^+S^- = 2(1-n_1) \tag{5}$$

$$S^-S^+ = 2(1-n_2). \tag{6}$$

Thus, from equation (3),

$$S^z = \tfrac{1}{2}[S^+, S^-] = n_2 - n_1 \tag{7}$$

in agreement with [12], and

$$(S^z)^2 = n_1 + n_2 - 2n_1n_2. \tag{8}$$

While equations (5) and (6) result in

$$\{S^+, S^-\} = 4 - 2n_1 - 2n_2 \tag{9}$$

equations (3) and (8) lead us to

$$\{S^+, S^-\} = 4 - 2n_1 - 2n_2 + 4n_1n_2. \tag{10}$$

The last addendum on the right-hand side of equation (10) is absent on the right-hand side of equation (9); that is, equations (9) and (10) contradict each other. Thus, the Batista–Ortiz transformations for an integer spin do not keep the spin operators algebra and hence are unacceptable in the form suggested in [12].

Now we turn to Batista–Ortiz transformations for half-odd integer spins. We consider the case of spin-3/2, for which in [12] it was offered to introduce three types of fermions. According to Batista–Ortiz transformations

$$S^+ = \sqrt{3}c_1^\dagger(1-n_2)(1-n_3) + 2c_2^\dagger c_1(1-n_3) + \sqrt{3}c_3^\dagger c_2(1-n_1) \tag{11}$$

and $S^- = (S^+)^\dagger$. Then

$$S^+S^- = 3n_1(1-n_2)(1-n_3) + 4n_2(1-n_1)(1-n_3) + 3n_3(1-n_2)(1-n_1) \tag{12}$$

$$S^-S^+ = 3(1-n_1)(1-n_2)(1-n_3) + 4n_1(1-n_2)(1-n_3) + 3n_2(1-n_1)(1-n_3). \tag{13}$$

So

$$\{S^+, S^-\} = 3 + 4n_1 + 4n_2 - 11n_1n_2 - 7n_2n_3 - 7n_1n_3 + 14n_1n_2n_3 \tag{14}$$

but

$$(S^z)^2 = \tfrac{1}{4}([S^+, S^-])^2 = \tfrac{9}{4} - 2n_1 - 2n_2 + \tfrac{7}{4}n_1n_2 - \tfrac{1}{4}n_2n_3 - \tfrac{1}{4}n_1n_3 + \tfrac{1}{2}n_1n_2n_3 \tag{15}$$

and we see that the third relation from equation (3) is not satisfied. Consequently, Batista–Ortiz transformations for half-odd integer spins are also wrong.

The general conclusion is that the Batista–Ortiz transformations are erroneous as Jordan–Wigner-type transformations. Also, their statement about the integrability of some spin-1 chains is not grounded.

However, transformations (1), (2) and (7) can be saved by adding to them the constraint

$$n_1n_2 = 0. \tag{16}$$

With this constraint, they may be used for representation of spin-1 through fermions. This is very important because no constraintless fermionic representations for integer spin exist (see Conclusion). However, equations (1), (2) and (7) are not Jordan–Wigner-type constraintless transformations.

Analogous remarks can be made for other formulae of Batista and Ortiz. But it is necessary to adduce the next observation. The minimum quantity of fermions required for fermionization (with constraint or not) of spin S is defined by the next rule. For fermionization of spin $S > 0$ satisfying the inequality

$$2^{n-1} < 2S + 1 \leqslant 2^n \qquad n \in N$$

n types of fermion are enough (this rule is the corollary of the fact that phase space of spin S is $(2S+1)$-dimensional and phase space of n types of fermion is 2^n-dimensional; this will become more clear after reading the whole of this letter). We see that this minimum quantity of particles for spin $S > 1$ is considerably less than $2S$ used in [12]. Thus, we can construct considerably more simple spin–fermion transformations for spin $S > 1$ using considerably less particles than in [12]. Furthermore, for some spins (specification will be given in the Conclusion) these transformations will be constraintless, i.e. of Jordan–Wigner type.

In the remaining part of this letter, we discuss only constraintless Jordan–Wigner-type transformations.

3. Fermionization of spin-3/2

In this section, we construct transformations between spin-3/2 operators and two types of fermion. From the procedure which will be used it will be clear what spins can be represented through fermions. First, we consider one-spin problem and after its solution generalize on a lattice.

The basis of the spin state space consists of four vectors, $|-3/2\rangle$, $|-1/2\rangle$, $|1/2\rangle$ and $|3/2\rangle$, which are the eigenvectors of the S^z operator. The phase space of two types of fermion is also four-dimensional with the basis $|0\rangle$, $|1\rangle = c_1^\dagger|0\rangle$, $|2\rangle = c_2^\dagger|0\rangle$, $|1\&2\rangle = c_2^\dagger c_1^\dagger|0\rangle$. Now, we need to establish a connection between spin operators and operators acting in fermionic space. Let us write out all the necessary matrix representations of these operators:

$$S^+ = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad S^z = \begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix} \tag{17}$$

and $S^- = (S^+)^\dagger = (S^+)^{\mathrm{T}}$;

$$c_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad c_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{18}$$

and $c_{1,2}^\dagger = (c_{1,2})^\dagger = (c_{1,2})^{\mathrm{T}}$. These matrices are simple enough and we can easily see that the solution of the one-spin problem is

$$\begin{cases} S^- = \sqrt{3}c_2 + 2c_2^\dagger c_1 \\ S^+ = \sqrt{3}c_2^\dagger + 2c_1^\dagger c_2. \end{cases} \tag{19}$$

It is easy to check that all the relations (3) with $S = \frac{3}{2}$ are satisfied by equation (19) with

$$S^z = \tfrac{1}{2}[S^+, S^-] = -\tfrac{3}{2} + 2n_1 + n_2. \tag{20}$$

By the passage to a lattice we must introduce the commutativity of all spin operators acting on different sites. Hence we must modify equation (19)

$$\begin{cases} S_i^- = \sqrt{3}c_{2i}U_i + 2c_{2i}^\dagger c_{1i} \\ S_i^+ = \sqrt{3}U_i^\dagger c_{2i}^\dagger + 2c_{1i}^\dagger c_{2i} \end{cases} \tag{21}$$

where, for a 1D lattice, $U_i = \exp\left\{\mathrm{i}\pi\sum_{j<i}(n_{1j} + n_{2j})\right\}$ is the string operator. The form (20) for S_i^z with site index is not changed.

The inverse transformations are obtained in the same way. For a lattice we have

$$\begin{cases} c_{1i} = -\frac{1}{\sqrt{3}}S_i^- S_i^z S_i^- W_i & c_{1i}^\dagger = -\frac{1}{\sqrt{3}}W_i^\dagger S_i^+ S_i^z S_i^+ \\ c_{2i} = \frac{1}{\sqrt{3}}\left(\frac{1}{2} + S_i^z\right)^2 S_i^- W_i & c_{2i}^\dagger = \frac{1}{\sqrt{3}}W_i^\dagger S_i^+\left(\frac{1}{2} + S_i^z\right)^2 \end{cases} \tag{22}$$

where, for a 1D lattice, $W_i = \prod_{j<i} X_j$ with

$$X_j = \frac{5}{4} - \left(S_j^z\right)^2 = \exp\left\{\mathrm{i}\frac{\pi}{2}\left(\frac{1}{4} - \left(S_j^z\right)^2\right)\right\}. \tag{23}$$

All the relations (4) for one site are fulfilled and the operator W_i ensures anticommutativity of c-operators acting on different sites. The form of X_j we can construct from the requirement

$$X_j c_{\alpha j} = -c_{\alpha j}X_j. \tag{24}$$

Assuming that X_j has a form

$$X_j = 1 + \xi S_j^z + \zeta \left(S_j^z\right)^2 + \eta \left(S_j^z\right)^3$$

with parameters ξ, ζ and η being defined from equation (24), taking into account equation (20) and introducing the normalization factor $\frac{5}{4}$, we obtain equation (23). Notice that $X_j = X_j^\dagger$ and $X_j^2 = 1$.

The obtained transformations (20), (21) and (22) are sufficiently complicated and much effort will be needed to derive some profit from them. Let us try to transform the isotropic XY spin-3/2 chain in fermionic form using equation (21):

$$\begin{aligned} H &= J \sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y\right) = \frac{1}{2} J \sum_i \left(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+\right) \\ &= \frac{1}{2} J \sum_i \left(3c_{2i}^\dagger (1 - 2n_{1i}) c_{2i+1} + 2\sqrt{3} U_i \left(c_{1i+1}^\dagger c_{2i+1} c_{2i} - c_{1i}^\dagger c_{2i} c_{2i+1}\right)\right. \\ &\quad \left. + \, 4c_{2i+1}^\dagger c_{1i+1} c_{1i}^\dagger c_{2i} + \text{h.c.}\right). \end{aligned} \tag{25}$$

We do not see the free fermion Hamiltonian as it was for the spin-1/2 case [6]. Now fermions interact, with the interaction being very complex due to operator U_i (string operator in the 1D case).

Notice that we can obtain usual Jordan–Wigner transformations using the procedure described in this section.

4. Conclusion

We have explicitly constructed direct (equations (20) and (21)) and inverse (equation (22)) Jordan–Wigner-type transformations between spin-3/2 and two types of fermion. These transformations can be used to map spin-3/2 systems on to fermionic models with two types of fermion, with spin up and with spin down, and vice versa, i.e. to map the fermionic model on to the spin model.

The answer to the second question raised in the introduction is still unclear and, perhaps, will appear after a time. However, concerning the first question, now we are ready to give the answer prompted by the procedure applied above to fermionize the spin-3/2. Only operators of spin S satisfying the equality

$$2S + 1 = 2^n \qquad n \in N \tag{26}$$

can be expressed in terms of fermions. The matrices of spin-S operators, which are the infinitesimal matrices of irreducible representation of $SU(2)$ group with the eigenvalue of the Casimir operator $\mathbf{S}^2$ being equal $S(S+1)$, have only the size $(2S+1) \times (2S+1)$ and, of course, cannot be presented through matrices of other sizes. The phase space of n types of fermion is 2^n-dimensional. The representation of the algebra of fermionic operators in this space is also irreducible. Hence, the condition (26) follows.

Spin 1 and the main part of other spins do not satisfy the condition (26). So, their operators cannot be expressed in terms of fermions (without constraint). The next spin suitable for fermionization is 7/2 with three types of fermion being necessary, then spin-15/2 follows with four types of fermion, etc. Of course, spin-1/2 satisfy the condition (26) with $n = 1$ and usual Jordan–Wigner transformations exist [3].

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